Advanced QFT @ EPFL 2024

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In QFT-I and -II we have seen canonical quantization based on nearly-free Lagrangians for particles of spin-0, 1/2 and 1 associated to scalars, spinors and vector fields, respectively:

(1)

Spin-0 $\longrightarrow$ scalar $\phi(x)$ $\quad S_{KG} = \int d^4x \, \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \ldots$

Spin-$\frac{1}{2}$ $\longrightarrow$ spinors:

- ψ_α = left-handed $\quad S_L = \int d^4x \, \psi^\dagger i\bar{\sigma}^\mu\partial_\mu\psi + \ldots$
- $\chi_{\dot\alpha}$ = right-handed $\quad S_R = \int d^4x \, \chi^\dagger i\sigma^\mu\partial_\mu\chi + \ldots$
- Ψ = Dirac $= \begin{pmatrix}\psi_\alpha \\ \chi^{\dot\alpha}\end{pmatrix}$ $\quad S_D = S_L + S_R - \int d^4x\, m\chi^\dagger\psi + \text{h.c.}$

spin-1 $\longrightarrow$ vector $A_\mu(x)$ $\quad S = \int d^4x \, -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \left(+\frac{m^2 A_\mu^2}{2}\right) + \ldots$

if massive

Hilbert space was obtained by diagonalizing a (nearly) harmonic Hamiltonian

Example:

(2) $$H_\phi = \int d^3x\, \mathcal{H} = \int d^3x\, \frac{1}{2}\left(\dot\phi^2 + (\vec\nabla\phi)^2 + m^2\phi^2\right) + \lambda\phi^4$$

$$= \int \frac{d^3k}{(2\pi)^3}\, \omega_k\, a_k^\dagger a_k + \text{const} + \text{anharmonics}$$

$$\phi = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(e^{-ikx}a(k) + e^{ikx}a^\dagger(k)\right) \quad \text{for free theory}$$

(3) $a(k)|0\rangle = 0$, $\quad |k\rangle \propto a^\dagger(k)|0\rangle$, $\quad |k_1,k_2\rangle \propto a^\dagger(k_1)a^\dagger(k_2)|0\rangle, \ldots$

(4) spin-0: $U(\Lambda)|k\rangle = |\Lambda k\rangle$, $\quad$ bosons: $|k_1 k_2\rangle = |k_2 k_1\rangle, \ldots$

The two major changes by working with ψ's and A_μ, as opposed to scalar fields, we have seen are

- ψ anticommute $\longrightarrow$ Fermionic oscillators

$$H = \sum_\lambda \int \frac{d^3k}{(2\pi)^3}\, \omega_k \left(a^\dagger a - b b^\dagger\right) \longrightarrow |k_1 k_2\rangle = -|k_2 k_1\rangle_{\cdots}$$

- $m=0$ $j=1$ $\longrightarrow$ Gauge theory : $A_\mu \sim A_\mu + \partial_\mu \Omega$ equivalent config.

(due to $2 \neq 4 \longrightarrow$ components of A_μ)
$\quad \llcorner$ d.o.f.

In canonical quantization à la Gupta-Bleuler one has an enlarged Hilbert sp.

$$\left(\begin{matrix}\text{Hilbert space}\\ \text{from canon. quant.}\end{matrix}\right) \supset \{|phys\rangle\} \sim \{|phys\rangle + L^\dagger(k)|phys\rangle\}$$

Quick Reminder:

$\{|phys\rangle\}$ is obtained by projecting on subspace solving one condition, $4\to 3$, and further reduced by identifying states within, $3 \to 2$.

Explicitly in Feynman's gauge:

$$S = S_{Maxwell} - \frac{1}{\xi}\int d^4x \left(\partial_\mu A^\mu\right)^2/2 \underset{\xi=1}{=} \int d^4x\, \frac{1}{2} A_\mu \Box A^\mu$$

$$[a_\mu(k), a^\dagger_\nu(p)] = -\eta_{\mu\nu}(2\pi)^3\delta^3(k-p)$$

$$\text{Hilbert sp.} = \text{span}\{a^\dagger(k_1)\dots a^\dagger(k_n)|0\rangle\} \supset \{|phys\rangle\}$$

$|phys\rangle$ iff $L(k)|phys\rangle = 0$ $\quad L(k) = k_\mu a^\mu(k)$ ("no $\partial_\mu A^\mu$-quanta")

$|phys\rangle \overset{2}{\sim} |phys\rangle + \int L^\dagger(p)|phys\rangle$ $\quad$ since $[L(k), L^\dagger(p)] \propto k^2 = 0$

e.g. $|0\rangle$ physical & equivalent to $(a^\dagger_0 - a^\dagger_3)|0\rangle$ for $k = \begin{pmatrix}\omega\\0\\0\\\omega\end{pmatrix}$

In the first few lectures we are going back to these fundamentals abstracting from any Lagrangian, perturbative, formulation, focusing on the non-perturbative role of symmetries + QM

— States —

The strategy is identifying the relevant group of symmetry so that the vector space where its (unitary and, possibly, projective) irreps act form the building block for generic (reducible) states, schematically

(5) $$|\text{state}\rangle = \bigoplus |(\text{proj.})\text{irreps symmetry}\rangle$$

stating sym. is the primary input, in this perspective

— Symmetry Group : Poincaré —

- Poincaré = $ISO(3,1)$ = Isometries of Minkowski space time

(6) $$x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu + a^\mu \qquad \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$\Lambda \in O(3,1)$ or subgroups such as $SO^\uparrow(3,1)$, $SO(3,1)$, $O^\uparrow(3,1)$

(typically only this one is actually respected by interact. as T & P are broken by weak inter.)

- Generators P_μ and $J_{\mu\nu}$

(7) $$[P_\mu, P_\nu] = 0 \qquad [J^{\alpha\beta}, P_\mu] = -i\,\delta_\mu^{[\alpha}\eta^{\beta]\nu} P_\nu \qquad [J^{\alpha\beta}, J_{\gamma\delta}] = +i\,\delta^{[\alpha}_{[\gamma}\eta^{\beta]\sigma} J_{\delta]\sigma}$$

$$\left[\!\left[\, U(\Lambda) = \exp\left(i\tfrac{\omega_{\mu\nu}}{2} J^{\mu\nu}\right);\quad U^{-1}(\Lambda) P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu \qquad U^{-1}(\Lambda) J^{\alpha\beta} U(\Lambda) = \Lambda^\alpha{}_\sigma \Lambda^\beta{}_\rho J^{\sigma\rho} \,\right]\!\right]$$

(8) $$W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$$ Pauli-Lubanski pseudo-vector

which, we remind, satisfies 3 important properties

(9) $$\begin{cases} P_\mu W^\mu = 0 & (a) \longrightarrow \text{only 3 indep.} \\ [P^\mu, W^\nu] = 0 & (b) \longrightarrow \text{leave fixed-}p^\mu \text{ space invariant} \\ [W^\mu, W^\nu] = -i\,\varepsilon^{\mu\nu\rho\sigma} W_\rho P_\sigma & (c) \longrightarrow \text{generate Little group } LG_p \end{cases}$$

- **Casimir Operators** $\quad P_\mu P^\mu = P^2 \qquad W_\mu W^\mu = W^2$

The irreps are labelled by Casimir operators P^2 & W^2 that commute with all group elements, thanks to Schur's (second) lemma

[$D(g): \mathcal{H} \to \mathcal{H}$ irrep of G and $\Lambda: \mathcal{H} \to \mathcal{H}$ such that $[D(g), \Lambda] = 0 \ \forall g \in G \Rightarrow \Lambda = \lambda \mathbb{1}$

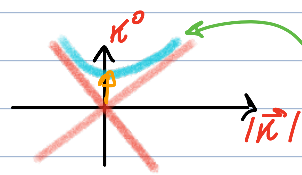
If $D(g)$ reducible $D(g) = \begin{pmatrix} D_1 & & \\ & D_2(g) & \\ & & \ddots \end{pmatrix}$ and $\Lambda = \begin{pmatrix} \lambda_1 \mathbb{1}_1 & & \\ & \lambda_2 \mathbb{1}_2 & \\ & & \ddots \end{pmatrix}$ so that λ_i label the D_i]

They are casimir op. because clearly $\begin{cases} [P_\mu, P^2] = 0 = [J_{\mu\nu}, P^2] \\ [P_\mu, W^2] = 0 = [J_{\mu\nu}, W^2] \end{cases}$ and

$U(a, \omega) = \exp(-i a_\mu P^\mu) \exp\left(\frac{i \omega_{\mu\nu}}{2} J^{\mu\nu}\right)$

The $\Lambda = P_\mu^2 = m^2 \mathbb{1}$ identify the mass of the irrep. $\Big\langle$ massive irreps / massless irreps

Massive Irreps: $\quad P^2 = m^2 > 0 \qquad P^0 > 0$



all this orbit same P^2, W^2

$[P^2, \text{Lorentz}] = 0 = [W^2, \text{Lorentz}] \Rightarrow$ Study them on rest frame

$P^\mu |k, a\rangle = k^\mu |k\, a\rangle$ with $k^\mu = \bar{k}^\mu = m\, \delta^\mu_0 = \begin{pmatrix} m \\ \vec{0} \end{pmatrix}$ in rest frame

$P^2 |k\, a\rangle = m^2 |k\, a\rangle$

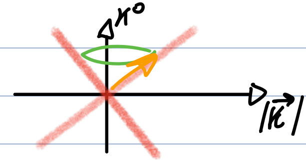
$W^\mu \Big|_{k=\bar{k}} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho}\, m\, \delta_\sigma^0 = \begin{cases} \mu = 0 & 0 \\ \mu = i & -m J^i \end{cases}$ angular mom. generator

(10) $\boxed{W_\mu^2 = -m^2 \vec{J}^2 = -m^2 j(j+1)}$ Irreps of $\vec{J}^2$ identified by $\boxed{j \in \frac{\mathbb{Z}}{2}}$

$|\bar{k}\ \sigma\rangle_{m^2, j}$ where $\sigma = -j, -j+1, \dots, +j$ $\quad$ $(2j+1)$-dimensional

Remark: This is perfectly consistent with (9c): for $k = \bar{k} = m\,\delta^0_\mu$ is $SU(2)$ algebra and indeed W_μ leave $\bar{k}$-space invariant by spanning $2j+1$ subspace.

Massless Irreps: $P^2=0$ $\quad P^0>0$



Can choose frame where $\bar{k}^\mu = \bar{E}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \bar{E}(\delta^\mu_0 + \delta^\mu_3)$

(11) $$W^\mu\Big|_{k=\bar{k}} = \begin{cases} J_3 \cdot \bar{E} \\ (J_1 - K_2)\cdot \bar{E} \\ (J_2 + K_1)\cdot \bar{E} \\ J_3 \cdot \bar{E} \end{cases}$$ by direct inspection plugging $\bar{k}^\mu$ in

where $\begin{cases} J_i = \frac{1}{2}\varepsilon_{ijk} J^{jk} & \text{rotat. gen.} \\ K_i = J_{i0} & \text{boosts gen.} \end{cases}$ $\qquad \varepsilon_{ijk} = \varepsilon_{0ijk}$

(minus signs maybe flying around...)

Rather than SU(2) algebra, the $W^\mu\big|_{k\text{-null}}$ generates ISO(2)

(12) $$\begin{cases} [W^1_{\bar{k}}, W^2_{\bar{k}}] = \bar{E}^2\Big(\overbrace{[J_1, J_2]}^{iJ_3} + \overbrace{[K_1, K_2]}^{-iJ_3}\Big) = 0 \\ [W^3_{\bar{k}}, W^1_{\bar{k}}] = \bar{E}^2\Big(\overbrace{[J_3, J_1]}^{iJ_2} - \overbrace{[J_3, K_2]}^{-iK_1}\Big) = i\bar{E}\, W^2_{\bar{k}} \\ [W^3_{\bar{k}}, W^2_{\bar{k}}] = \bar{E}^2\Big(\underbrace{[J_3, J_2]}_{-iJ_1} + \underbrace{[J_3, K_1]}_{iK_2}\Big) = -i\bar{E}\, W^1_{\bar{k}} \end{cases}$$

The W_μ^2 cannot be positive

(13) $$W_\mu^2\Big|_{\bar{k}} = \cancel{(W^0|_{\bar{k}})^2} - \cancel{(W^3|_{\bar{k}})^2} - (W^1|_{\bar{k}})^2 - (W^2|_{\bar{k}})^2 = -(W_1 + iW_2)_{\bar{k}}(W_1 - iW_2)_{\bar{k}}$$

$$= -W_-^\dagger W_-\, \bar{E}^2 \le 0$$

where we changed basis for future convenience

(14) $$W_\pm \equiv (W^1_{\bar{k}} \pm i W^2_{\bar{k}})/\bar{E}\,, \qquad \mathbb{H} = W^0_{\bar{k}}/\bar{E}$$

(15) $$[\mathbb{H}, W_\pm] = \pm W_\pm \qquad [W_+, W_-] = 0 \qquad W_+^\dagger = W_-$$

raising / lowering

There are therefore 2 subcases for $m^2=0$

$$m^2=0:\begin{cases}(a)\ \ W_\mu^2<0 \quad W_+,W_-\neq 0 \longrightarrow \text{all } ISO(2) \text{ non-trivial}\\ (b)\ \ W_\mu^2=0 \quad W_+=W_-=0 \longrightarrow \text{only } U(1)\subset ISO(2) \text{ non trivial}\end{cases}$$

Case (a) $W_\mu^2=-\mu^2<0 \longrightarrow$ Infinite dim. irrep.

$$|\bar{\kappa}\ \lambda{-}1\rangle \underset{W_-}{\overset{W_+}{\rightleftarrows}} |\bar{\kappa}\ \lambda\rangle \underset{W_-}{\overset{W_+}{\rightleftarrows}} |\bar{\kappa}\ \lambda{+}1\rangle \underset{W_-}{\overset{W_+}{\rightleftarrows}} |\bar{\kappa}\ \lambda{+}2\rangle \qquad (16)$$

which are known as Continuous-Spin particles in the literature (often another basis is $|\bar{\kappa}\ \theta\rangle=\sum_\lambda e^{i\lambda\theta}|\bar{\kappa}\ \lambda\rangle$ where one diagon. $W_\pm$ simultaneously instead.)

[[states within irrep labelled by $\mathbb{H}$ eigenv. $\mathbb{H}|\bar{\kappa}\ \lambda\rangle=\lambda|\bar{\kappa}\ \lambda\rangle$
$\mathbb{H}W_\pm|\bar{\kappa}\ \lambda\rangle=W_\pm(\mathbb{H}\pm1)|\bar{\kappa}\ \lambda\rangle=(\lambda\pm1)W_\pm|\bar{\kappa}\ \lambda\rangle$ and $W_\mp|\kappa\ \lambda\rangle\neq0$ since $W_-W_+\neq0$]]

We will not study this case further. (For recent progress on it see 2406.17017)

Case (b) $W_+=W_-=0 \quad \mathbb{H}|\bar{\kappa}\ \lambda\rangle=\lambda|\bar{\kappa}\ \lambda\rangle$ $\quad\lambda$: Helicity

- $\mathbb{H}$ generates rotations $U(1)$ around $\vec{\kappa}$. ($W^3\propto J^3$)
- $\lambda\in\mathbb{Z}/2$ since $U(1)\subset$ Lorentz which is doubly connected
- λ Lorentz-invariant since $W_\mu=\lambda P_\mu$ or $\dfrac{\vec{J}\cdot\vec{P}}{|\vec{P}|}=\lambda$
- irreps 1-dim. ($U(1)$ is abelian)
- CPT : $|\bar{\kappa}\ \lambda\ q\rangle \longrightarrow |\bar{\kappa}\ {-\lambda}\ \bar{q}\rangle$ (since $U^{-1}(CPT)\,\vec{J}\,U(CPT)=-\vec{J}$, $U^{-1}(CPT)\,\vec{P}\,U(CPT)=+\vec{P}$)

internal charges

Summery so far

$m \neq 0 \qquad |\bar{k}\ \sigma\rangle_{m\,j} \xrightarrow{\Lambda \in LG_{\bar{k}} = SU(2)} |\bar{k}\ \sigma'\rangle_{m\,j}\, R^{(j)}_{\sigma'\sigma}$ (2j+1) dim irrep of SU(2)

$m = 0 \qquad W_\mu^2 = 0 \quad |\bar{k}\ \lambda\rangle \xrightarrow{\Lambda \in LG_{\bar{k}} = U(1)} |\bar{k}\ \lambda\rangle\, e^{-i\theta\lambda}$ 1-dim U(1) irrep

$\left(m = 0 \quad W_\mu^2 = -\mu^2 < 0 \quad |\bar{k}\ \lambda\rangle \xrightarrow{\Lambda \in LG_{\bar{k}} = ISO(2)} |\bar{k}\ \lambda'\rangle\, J_{\lambda\lambda'}(2|a|\mu)\, e^{-i\theta\lambda}\right)$ infinite dimens.

General Poincaré irreps

We know how the irreps act on a particular subspace (fixed-$\bar{k}$ momentum where $\bar{k}^\mu$ was a certain reference vector). On the other hand we can connect $\bar{k}^\mu \to k^\mu$ belonging to the same hyperboloid ($m \neq 0$) or lightcone ($m=0$) by a Lorentz transformation:

(17) $\quad L(k,\bar{k}) : \bar{k}^\mu \to L^\mu{}_\nu(k,\bar{k})\,\bar{k}^\nu = k^\mu$ (not the same L for $m=0$ or $m \neq 0$)

(not unique choice! Any $LG_k \circ L(k,\bar{k}) \circ LG_{\bar{k}}$ does the job)

Example: $L(k,\bar{k}) = R_{\theta\varphi}\, B_z(\eta)$ $\quad B_z \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E \\ 0 \\ 0 \\ \kappa \end{pmatrix} \quad R_{\theta\varphi} \begin{pmatrix} E \\ 0 \\ 0 \\ \kappa \end{pmatrix} = \begin{pmatrix} E \\ \vec{k} \end{pmatrix}$

From this, we define

(18) $$\begin{cases} |k\ \sigma\rangle_{m\,j} \equiv U(L(k,\bar{k}))\,|\bar{k}\ \sigma\rangle_{m\,j} \\ |k\ \lambda\rangle \equiv U(L(k,\bar{k}))\,|\bar{k}\ \lambda\rangle \end{cases}$$

(sometimes we will not display m j, using just σ to distinguish from m=0 case which has λ label instead) so that under a generic Lorentz Λ:

$$(19) \quad U(\Lambda)|k\,\sigma\rangle_{mj} = U(L(\Lambda k,\bar{k}))\, U(\underbrace{\overbrace{L^{-1}(\Lambda k,\bar{k})\Lambda L(k,\bar{k})}^{\text{''Wigner rot } W\text{''}}}_{\bar{k}\to k\to\Lambda k\to\bar{k}})|\bar{k}\,\sigma\rangle$$

$$= U(L(\Lambda k,\bar{k}))|\bar{k}\,\sigma'\rangle R^{(j)}_{\sigma'\sigma}(\Lambda,k,\bar{k})$$

Mutatis mutandi for $m=0$, we get

$$(20) \quad \boxed{\begin{aligned} U(\Lambda)|k\,\sigma\rangle_{mj} &= |\Lambda k\,\sigma'\rangle_{mj}\, R^{(j)}_{\sigma'\sigma}(W) \\ U(\Lambda)|k\,\lambda\rangle_{m=0} &= |\Lambda k\,\lambda\rangle\, e^{-i\lambda\theta(\Lambda,k,\bar{k})} \end{aligned}}$$

— Fields —

Along with the Poincaré irreps above, it's interesting to study the local excitations produced by acting with local operators – fields $\vartheta(x)$ – on $|0\rangle$, and see their relation to the Poincaré irreps.

Abstracting from the Lagrangian formulation, we are going to assume the Axioms

I. "Empty space exists, and looks the same to all inertial observers"

(which act for non-inertial via Unruh effect)

$$(21) \quad \exists\, |0\rangle \qquad \boxed{P_\mu|0\rangle = 0 = J_{\mu\nu}|0\rangle}$$

I'. "Spectrum condition & stability"

$$(22) \quad \boxed{P^0 \geq 0 \qquad P_\mu^2 \geq 0}$$

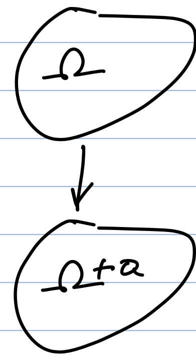
Comment: I. and I'. are not specific to field theory yet.

II. "Can create local excitations, and reproduce them around"

(23) $\quad \exists \quad \vartheta(x), \qquad \int_{\Omega} f(x)\, \vartheta(x)\, |0\rangle = |f_\Omega\rangle \quad \sim$

Ω ← support of f

(24) $\quad \vartheta(x+a) = \exp(i a_\mu P^\mu)\, \vartheta(x)\, \exp(-i a_\nu P^\nu)$



Remark: One typically assumes a stronger condition, namely any state can be reached by inserting sufficiently many local operators, i.e.

Hilbert Space = span by $\int f(x_1) \ldots f(x_n)\, \vartheta(x_1) \ldots \vartheta(x_n)\, |0\rangle$

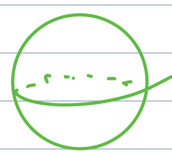
III. "Lorentz: fields carry Lorentz (projective) irreps"

(25) $$\boxed{\vartheta_A(x) \longrightarrow U^{-1}(\Lambda)\, \vartheta_A(x)\, U(\Lambda) = D_{AB}(A)\, \vartheta_B(\Lambda^{-1} x)}$$

$D_{AB}(A)$ is a $SL(2,\mathbb{C})$-irrep $\qquad \Lambda^\mu{}_\nu = \Lambda^\mu{}_\nu(A) \qquad A \in SL(2,\mathbb{C})$

Reminder: $\hat{SO}(3,1) = SL(2,\mathbb{C})/\mathbb{Z}_2$ via 2-to-1 homomorphism (see QFT-II for details)

$$\underset{SL(2,\mathbb{C})}{\underset{\cap}{A}}\, \sigma_\mu A^\dagger = \sigma_\nu \Lambda^\nu{}_\mu(A) \xrightarrow[\mathrm{Tr}(\bar\sigma^\mu \sigma_\nu) = 2\delta^\mu_\nu]{} \Lambda^\mu{}_\nu(A) = \frac{1}{2} \mathrm{Tr}(\bar\sigma^\mu A \sigma_\nu A^\dagger)$$

$SL(2,\mathbb{C}) \sim S_3 \times \mathbb{R}^3 =$  $\times\ \mathbb{R}^3 \Rightarrow$ simply connected

$\hat{SO}(3,1) \sim S_3/\mathbb{Z}_2 \times \mathbb{R}^3 =$  identified $\times\ \mathbb{R}^3$

closed path not contractible to point

Irreps of $SL(2,\mathbb{C})$ provide the projective irreps of $\hat{SO}(3,1)$. (later I will call the $A \equiv \Lambda_L \in SL(2,\mathbb{C})$)

Examples

(26) vector $\quad \vartheta_\mu(x) \longrightarrow \Lambda_\mu{}^\nu\, \vartheta_\nu(\Lambda^{-1}x)$

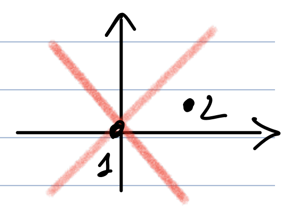
(27) left-handed spinor $\quad \vartheta_\alpha(x) \longrightarrow \begin{cases} A_\alpha{}^\beta\, \vartheta_\beta(\Lambda^{-1}(A)\,x) \\ A_\alpha{}^\beta = \exp\left(-i(\vec\theta - i\vec\eta)\frac{\vec\sigma}{2}\right)_\alpha{}^\beta \end{cases}$

(28) right-handed spinor $\quad \vartheta_{\dot\alpha}(x) \longrightarrow \begin{cases} B_{\dot\alpha}{}^{\dot\beta}\, \vartheta_{\dot\beta}(\Lambda^{-1}(A)\,x) \\ B_{\dot\alpha}{}^{\dot\beta} = (A_\alpha{}^\beta)^* \end{cases}$

Remark: For complexified Lorentz $so(3,1)_{\mathbb{C}} = sl^L(2,\mathbb{C})\times sl^R(2,\mathbb{C})/\mathbb{Z}_2$ and A & B are two independent matrices in $A\in sl^L(2,\mathbb{C})$ $\;B\in sl^R(2,\mathbb{C})$

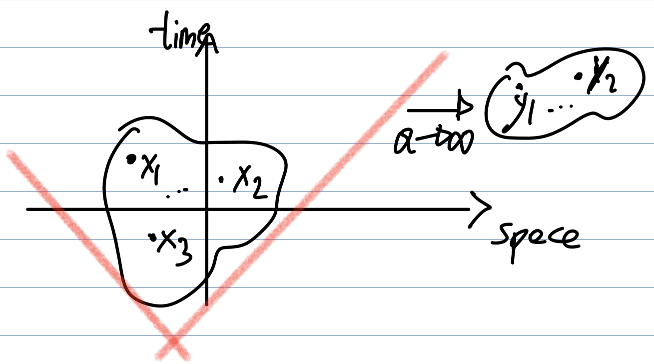
IV. "Causality"

(29) $$\boxed{[\vartheta(x_1),\vartheta(x_2)]=0 \quad \text{if} \quad (x_1-x_2)^2<0}$$



V. "Cluster decomposition"

(30) $$\boxed{\begin{aligned}&\langle 0|\,\vartheta(x_1)\ldots\vartheta(x_n)\,\vartheta(y_1+a)\ldots\vartheta(y_m+a)|0\rangle \\ &\xrightarrow[\text{spacelike}]{a\to\infty} \langle 0|\,\vartheta(x_1)\ldots\vartheta(x_n)|0\rangle\,\langle 0|\,\vartheta(y_1)\ldots\vartheta(y_m)|0\rangle\end{aligned}}$$



This says that correlations dye off at spacelike infinity.
In a sense is the statement that vacuum is not entangled

Remark: This basically implies vacuum is unique, since we can consider the 2pt-function

(31) $\langle 0|\,\mathcal{O}(x)\,\mathcal{O}(0)|0\rangle_\lambda$ with $x^\mu = (0, \vec{x})$ spacelike

$$\mathbb{1} = \Sigma_i |0\rangle_i\, {}_i\langle 0| + \sum_N \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}|p\,N\rangle\langle p\,N|$$

← all other quantum numbers but p

so that using translation invariance

(32) $$\underbrace{\Sigma_i \langle 0|\,\mathcal{O}(0)|0\rangle_i\, {}_i\langle 0|\mathcal{O}(0)|0\rangle}_{\vec{x}\text{- independent}} + \underbrace{\sum \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p} e^{+i\vec{p}\cdot\vec{x}} \langle 0|\mathcal{O}(0)|pN\rangle\langle pN|\mathcal{O}(0)|0\rangle}_{\text{Fourier transform} \xrightarrow[|\vec{x}|\to\infty]{} 0 \text{ (under suitable regularity)}}$$

$\Rightarrow$ fails factorization $\to \langle 0|\mathcal{O}|0\rangle\langle 0|\mathcal{O}|0\rangle$ unless $|0\rangle_i = |0\rangle$ only.

Comment: The 5 axioms above will be further extended in future lectures to include the existence of asymptotic states associated to particles. We will use cluster decomposition again in scattering theory to derive LSZ.

— Lorentz & $SL(2,\mathbb{C})$ —

Now that we have stated our assumptions we will start deriving general universal results by inspecting the relation between fields & states. To this goal, however, is convenient to recall how Lorentz (really $SL(2,\mathbb{C})$) irreps are constructed by exponentiating the (complexified) Lie algebras

(33) $$\boxed{[J^i, J^j] = i\varepsilon^{ijk} J^k \quad [J^i, K^j] = i\varepsilon^{ijk} K^k \quad [K^i, K^j] = -i\varepsilon^{ijk} J^k}$$ Lorentz algebra

for which a better (complexified) basis is

(34) $$J^i_\pm = (J^i \pm i K^i)/2 \longleftrightarrow \vec{J}_+ + \vec{J}_- = \vec{J} \quad \frac{\vec{J}_+ - \vec{J}_-}{i} = \vec{K}$$

(35) $$\boxed{[J^i_+, J^j_+] = i\varepsilon^{ijk} J^k_+ \quad [J^i_-, J^j_-] = i\varepsilon^{ijk} J^k_- \quad [J^i_+, J^j_-] = 0}$$

two copies of $su(2)$-algebra: $so(3,1)_{\mathbb{C}} = sl(2,\mathbb{C})_{\mathbb{C}} = su(2) \oplus_{\mathbb{C}} su(2)$

Comment: exponentiating the complexified $su(2) \oplus_{\mathbb{C}} su(2)$ algebra we get complexified $SL(2,\mathbb{C})_{\mathbb{C}} = SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C})$. To get just $SL(2,\mathbb{C})$ one needs to exponentiate the algebra with some restrictions on the Lie parameters, such that J^i and K^i will have real coefficients. Explicitly, $\alpha_i J^i + \beta_i K^i = (\alpha_i - \beta_i i) J^i_+ + (\alpha_i + \beta_i i) J^i_- = \xi_{+i} J^i_+ + \xi_{-i} J^i_-$ so that $SL(2,\mathbb{C})$ is obtained by $(\alpha_i, \beta_i) \in \mathbb{R}^6$ that is $\underline{\xi^*_{+i} = \xi_{-i}}$. If one enforces a different constraint, such as e.g. $(\xi_+, \xi_-) \in \mathbb{R}^6$, one gets instead $SO(4)$. The full $SL(2,\mathbb{C})_{\mathbb{C}}$ comes from $(\xi_+, \xi_-) \in \mathbb{C}^6$, i.e. unconstr. We also remind that the explicit homomorphism $SO(3,1)_{\mathbb{C}} \sim SL(2,\mathbb{C})_{\mathbb{C}} / \mathbb{Z}_2$ comes about from observing $A^{-1}\sigma^\mu B = \Lambda^\mu{}_\nu(A,B)\sigma^\nu$ where $(A,B) \in SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ i.e. $\Lambda^\mu{}_\nu(A,B) = \mathrm{Tr}(\bar{\sigma}_\nu A^{-1} \sigma^\mu B)/2$. This follows directly from observing that $\det(\sigma^\mu x_\mu) = x_\mu^2$ is left invariant by $\sigma^\mu x_\mu \to A \sigma^\mu x_\mu B^{-1}$ since $\det A = 1 = \det B$, even for $x_\mu \in \mathbb{C}^4$. See QFT-II notes for details

The irreps obtained by exponentiating $su(2)\oplus_{\mathbb{C}} su(2)$ (with or without restrictions on Lie parameters) are thus identified by a pair of half-integer numbers

(36) $\quad (j_-, j_+) \in \frac{\mathbb{N}}{2} \times \frac{\mathbb{N}}{2} \longrightarrow \boxed{\vartheta^{(j_-,j_+)}_{(0)} \quad (2j_-+1)\times(2j_++1) \text{ dimensional}}$

Example:

$$\boxed{\left(\tfrac{1}{2},0\right):\ \vartheta^{(1/2,0)}_{\alpha\,(0)} \qquad \vec{J} = \vec{J}_+ = \frac{\vec{\sigma}}{2} \qquad \vec{K} = -i\vec{J}_+ = -i\,\frac{\vec{\sigma}}{2}}$$

α: 2 values

$$\exp\left(-i\left(\theta^i\frac{\sigma^i}{2} - i\eta^i\frac{\sigma^i}{2}\right)\right)_\alpha{}^\beta\, \vartheta^{(1/2,0)}_{\beta\,(0)} = (\Lambda_L)_\alpha{}^\beta\, \vartheta^{(1/2,0)}_{\beta\,(0)}$$

(what above I called A) $\quad \Lambda_L \in SL(2,\mathbb{C})$ since $Tr(\sigma^i)=0$

$$\boxed{(0,\tfrac{1}{2}):\ \vartheta^{\dot\alpha\,(0,1/2)}_{(0)}} \longrightarrow \exp\left(-i\left(\theta^i\frac{\sigma^i}{2} + i\eta^i\frac{\sigma^i}{2}\right)\right)^{\dot\alpha}{}_{\dot\beta}\, \vartheta^{\dot\beta\,(0,1/2)}_{(0)}$$

$$= (\Lambda_R)^{\dot\alpha}{}_{\dot\beta}\, \vartheta^{\dot\beta\,(0,1/2)}_{(0)}$$

$\Lambda_R \in SL(2,\mathbb{C})$ $\qquad$ (Λ_R was called $(B^{-1})^T$ above)

Remarks:

- $\Lambda_R{}^{\dot\alpha}{}_{\dot\beta}$ and $\Lambda_L{}^\alpha{}_\beta$ are related, $\Lambda_R^\dagger = \Lambda_L^{-1}$ & $\Lambda_R^{-1} = \Lambda_L^\dagger$

  (carefull: for complexified-$SL(2,\mathbb{C})$ they are not)

  This is consistent with (28) because there we can raise/lower indices with the invariant tensors $\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$, $\varepsilon_{\dot\alpha\dot\beta}$, $\varepsilon^{\dot\alpha\dot\beta}$

(37) $\vartheta_{\dot\alpha}^{(0,1/2)}(0) \equiv \varepsilon_{\dot\alpha\dot\beta}\, \vartheta^{\dot\beta\,(0,1/2)}(0) \longrightarrow \varepsilon_{\dot\alpha\dot\beta}\, (\Lambda_R)^{\dot\beta}{}_{\dot\gamma}\, \vartheta^{\dot\gamma\,(0,1/2)}(0)$

$$= \varepsilon_{\dot\alpha\dot\beta}\, \Lambda_R{}^{\dot\beta}{}_{\dot\gamma}\, \varepsilon^{\dot\gamma\dot\sigma}\, \vartheta_{\dot\sigma}^{(0,1/2)}(0) = (\Lambda_L{}_{\alpha}{}^{\sigma})^*\, \vartheta_{\dot\sigma}^{(0,1/2)}(0)$$

$\sigma^2 \Lambda_R \sigma^2 = (\Lambda_R^{-1})^T = \Lambda_L^*$

$\sigma^2 \vec\sigma\, \sigma^2 = -\vec\sigma^T$

i.e. way of saying that

(38). $\boxed{(1/2, 0) \overset{*}{\longleftrightarrow} (0, 1/2)}$

(again, not for $SL(2,\mathbb{C})_{\mathbb{C}} \sim SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$)

Example 4-vector

$\boxed{(1/2, 1/2):\ \vartheta_{\alpha\dot\alpha}^{(1/2,1/2)}(0)}$ Transforms just as tensor product of left- and right

$\downarrow$

(39) $(\Lambda_L)_\alpha{}^\beta (\Lambda_R)_{\dot\alpha}{}^{\dot\beta}\, \vartheta_{\beta\dot\beta}^{(1/2,1/2)}(0) = (\Lambda_L)_\alpha{}^\beta (\Lambda_L^*)_{\dot\alpha}{}^{\dot\beta}\, \vartheta_{\beta\dot\beta}^{(1/2,1/2)}(0)$

or in index-free notation

(40) $\vartheta^{(1/2,1/2)}(0) \longrightarrow \Lambda_L\, \vartheta^{(1/2,1/2)}(0)\, \Lambda_L^\dagger$

The $(1/2, 1/2)$-irrep is nothing but the 4-vector irrep

(41) $\begin{cases} \vartheta_\mu(0)\, \sigma^\mu_{\alpha\dot\alpha} \longrightarrow \vartheta_\nu\, \Lambda^\nu{}_\mu\, \sigma^\mu_{\alpha\dot\alpha} \underset{SL(2,\mathbb{C})/\mathbb{Z}_2 = SO(3,1)}{=} \left[\Lambda_L (\vartheta_\nu \sigma^\nu) \Lambda_L^\dagger\right]_{\alpha\dot\alpha} = (1/2, 1/2) \\ \frac{1}{2}\bar\sigma^{\mu\dot\alpha\alpha}\, \vartheta_{\alpha\dot\alpha}^{(1/2,1/2)}(0) = \mathrm{Tr}(\bar\sigma^\mu \vartheta_{(0)}^{(1/2,1/2)})/2 \longrightarrow \mathrm{Tr}(\Lambda_L^\dagger \bar\sigma^\mu \Lambda_L \vartheta_{(0)}^{(1/2,1/2)}) = \Lambda^\mu{}_\nu(\Lambda_L)\,\mathrm{Tr}(\ldots^\nu\ldots) \end{cases}$

Since in general

$$(42)\quad \begin{cases} (N,0) \subset \overbrace{(\tfrac12,0)\otimes\cdots\otimes(\tfrac12,0)}^{2N \text{ times}} & \text{by taking the fully sym. comb.} \\ (0,N) \subset (0,\tfrac12)\otimes\cdots\otimes(0,\tfrac12) & (\underbrace{\uparrow\uparrow\uparrow\uparrow\uparrow\ldots\uparrow}_{N} \text{ has "spin" } N/2) \end{cases}$$

$\Downarrow$

General $SL(2,\mathbb{C})$ irrep (j_-, j_+):

$$(43)\quad \boxed{\begin{gathered} \vartheta^{(j_-,j_+)}_{\underbrace{\alpha_1\ldots\alpha_{2j_-}}_{\text{fully sym}}\;\underbrace{\dot\alpha_1\ldots\dot\alpha_{2j_+}}_{\text{fully sym}}}(0) \longrightarrow (\Lambda_L)_{\alpha_1}{}^{\beta_1}\cdots(\Lambda_L)_{\alpha_{2j_-}}{}^{\beta_{2j_-}}(\Lambda_R)_{\dot\alpha_1}{}^{\dot\beta_1}\cdots(\Lambda_R)_{\dot\alpha_{2j_+}}{}^{\dot\beta_{2j_+}}\,\vartheta^{(j_-,j_+)}_{\beta_1\ldots\beta_{2j_-}\dot\beta_1\ldots\dot\beta_{2j_+}}(0) \\ \text{with } \Lambda_L \in SL(2,\mathbb{C}) \text{ and } \Lambda_R^* = \Lambda_L \text{ (if } SL(2,\mathbb{C}) \text{ not complexified)} \end{gathered}}$$

Remarks

- a $\vartheta_{\alpha_1\ldots\alpha_n\dot\alpha_1\ldots\dot\alpha_m}$ not fully antisymmetric in α's and $\dot\alpha$'s would be reducible ($\vartheta_{\ldots\alpha\beta\ldots}\varepsilon^{\alpha\beta} \neq 0$ in that case and since $\varepsilon^{\alpha\beta}$ invariant tensor there would be a proper invariant subspace $\vartheta_{\ldots\alpha\beta\ldots} - \vartheta_{\ldots\beta\alpha\ldots} \propto \varepsilon_{\alpha\beta}\,\vartheta_{\ldots\rho\sigma\ldots}\varepsilon^{\rho\sigma}$)
- Lorentz tensors are those for which $\vec{J}$ is integer $\Rightarrow j_- + j_+$ integer

One can go to a manifest Lorentz-index basis by contracting with Pauli matrices: e.g. for an antisymmetric $F_{\mu\nu} = -F_{\nu\mu}$

$$(44)\quad \frac14 F_{\mu\nu}\,\sigma^{[\mu}_{\alpha\dot\alpha}\,\sigma^{\nu]}_{\beta\dot\beta}\,\varepsilon^{\dot\alpha\dot\beta} \equiv \mathbb{F}_{\alpha\beta} = \mathbb{F}_{\beta\alpha} \Rightarrow (1,0)$$

$$(45)\quad \frac14 F_{\mu\nu}\,\sigma^{[\mu}_{\alpha\dot\alpha}\,\sigma^{\nu]}_{\beta\dot\beta}\,\varepsilon^{\beta\alpha} \equiv \tilde{\mathbb{F}}_{\dot\alpha\dot\beta} = \tilde{\mathbb{F}}_{\dot\beta\dot\alpha} \Rightarrow (0,1)$$

$F_{\mu\nu} = (1,0) \oplus (0,1)$

(in general reducible, can be written: $\mathbb{F}_{\alpha\beta\dot\alpha\dot\beta} = \frac12 \mathbb{F}_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta} + \frac12 \tilde{\mathbb{F}}_{\dot\alpha\dot\beta}\,\varepsilon_{\alpha\beta}$)

- Parity exchanges left $\leftrightarrow$ right since $\begin{cases} J_i \xrightarrow{P} J_i \\ K_i \to -K_i \end{cases}$

(46) $$\begin{cases} P: \quad (j_-, j_+) \longleftrightarrow (j_+, j_-) \\ U(P)^{-1}\, \vartheta_A^{(j_-,j_+)}(0)\, U(P) = \mathbb{P}_{A\bar{B}}\, \vartheta_{\bar{B}}^{(j_+,j_-)}(0) \end{cases} \qquad \mathbb{P}_{A\bar{B}} \mathbb{P}_{\bar{B}C} = \delta_{AC}$$

collective index

not closed for chiral multiplets.